

# ***L*-COMPLETE HOPF ALGEBROIDS AND THEIR COMODULES**

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**ABSTRACT.** We investigate Hopf algebroids in the category of *L*-complete modules over a commutative Noetherian regular complete local ring. The main examples of interest in algebraic topology are the Hopf algebroids associated to Lubin-Tate spectra in the  $K(n)$ -local stable homotopy category, and we show that these have Landweber filtrations for all finitely generated discrete modules.

Along the way we investigate the canonical Hopf algebras associated to Hopf algebroids over fields and introduce a notion of unipotent Hopf algebroid generalising that for Hopf algebras.

## INTRODUCTION

In this paper we describe some algebraic machinery that has been found useful occurs when working with the  $K(n)$ -local homotopy category, and specifically the cooperation structure on covariant functors of the form  $E_*^\vee(-)$ , where

$$E_*^\vee(X) = \pi_*(L_{K(n)}(E \wedge X))$$

is the homotopy of the Bousfield localisation of  $E \wedge X$  with respect to Morava  $K$ -theory  $K(n)$ . Our main focus is on algebra, but our principal examples originate in stable homotopy theory.

In studying the  $K(n)$ -local homotopy category, topologists have found it helpful to use the notion of *L-complete module* introduced for other purposes by Greenlees and May in [5]. It is particularly fortunate that the Lubin-Tate spectrum  $E_n$  associated with a prime  $p$  and  $n \geq 1$  has for its homotopy ring

$$\pi_* E_n = W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]] [u, u^{-1}],$$

where all generators are in degree 0 except  $u$  which has degree 2. Thus (apart from the odd-even grading) the coefficient ring for the covariant functor  $(E_n)_*^\vee(-)$  is a commutative Noetherian regular complete local ring of dimension  $n$  and the theory of *L*-complete modules works well. For details of these applications see [7], also [6] and [2, section 7].

In this paper we consider analogues of Hopf algebroids in the category of *L*-complete modules over a commutative Noetherian regular complete local ring, and relate this to our earlier work of [1]. Since the latter appeared there has been a considerable amount of work by Hovey and Strickland [8, 9] on localisations of categories of comodules over  $MU_*MU$  and  $BP_*BP$ , but that seems to be unrelated to the present theory. One of our main motivations was to try to

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This paper is dedicated to the mountains of the Arolla valley.

understand the precise sense in which the  $L$ -complete theory differs from these other theories and we intend to return to this in future work.

We introduce a notion of unipotent Hopf algebroid over a field and then consider the relationship between modules over a Hopf algebroid  $(\mathbf{k}, \Gamma)$  and over its associated Hopf algebra  $(\mathbf{k}, \Gamma')$  and unicursal Hopf algebroid. We show that if  $(\mathbf{k}, \Gamma')$  is unipotent then so is  $(\mathbf{k}, \Gamma)$ . As a consequence a large class of Hopf algebroids over local rings have composition series for finitely generated comodules which are discrete in the sense that they are annihilated by some power of the maximal ideal.

We end by discussing the important case  $E_*^\vee E$  for a Lubin-Tate spectrum  $E$ . In particular we verify that finitely generated comodules over this  $L$ -complete Hopf algebroid have Landweber filtrations.

For completeness, in two appendices we continue the discussion of the connections with twisted group rings begun in [1], and expand on a result of [6] on the non-exactness of coproducts of  $L$ -complete modules.

## 1. $L$ -COMPLETE MODULES

Let  $(R, \mathfrak{m})$  be a commutative Noetherian regular local ring, and let  $n = \dim R$ . We denote the category of (left)  $R$ -modules by  $\mathcal{M} = \mathcal{M}_R$ . Undecorated tensor products will be taken over  $R$ , i.e.,  $\otimes = \otimes_R$ . We will often write  $\widehat{R}$  for the  $\mathfrak{m}$ -adic completion  $R_\mathfrak{m}$ , and  $\widehat{\mathfrak{m}}$  for  $\mathfrak{m}_\mathfrak{m}$ .

The  $\mathfrak{m}$ -adic completion functor

$$M \mapsto M_\mathfrak{m}^\wedge$$

on  $\mathcal{M}$  is neither left nor right exact. Following [5], we consider its left derived functors  $L_s = L_s^\mathfrak{m}$  ( $s \geq 0$ ). We recall that there are natural transformations

$$\text{Id} \xrightarrow{\eta} L_0 \longrightarrow (-)_\mathfrak{m}^\wedge \longrightarrow R/\mathfrak{m} \otimes_R (-).$$

The two right hand natural transformations are epimorphic for each module, and  $L_0$  is idempotent, i.e.,  $L_0^2 \cong L_0$ . It is also true that  $L_s$  is trivial for  $s > n$ . For computing the derived functor for an  $R$ -module  $M$  and  $s > 0$  there is a natural exact sequence of [5, proposition 1.1]:

$$(1.1) \quad 0 \rightarrow \lim_k^1 \text{Tor}_{s+1}^R(R/\mathfrak{m}^k, M) \longrightarrow L_s M \longrightarrow \lim_k \text{Tor}_s^R(R/\mathfrak{m}^k, M) \rightarrow 0.$$

It is an important fact that tensoring with finitely generated modules interacts well with the functor  $L_0$ . A module is said to have *bounded  $\mathfrak{m}$ -torsion module* if it is annihilated by some power of  $\mathfrak{m}$ .

**Proposition 1.1.** *Let  $M, N$  be  $R$ -modules with  $M$  finitely generated. Then there is a natural isomorphism*

$$M \otimes L_0 N \longrightarrow L_0(M \otimes N).$$

*In particular,*

$$\begin{aligned} L_0 M &\cong M_\mathfrak{m}^\wedge \cong \widehat{R} \otimes M, \\ R/\mathfrak{m}^k \otimes L_0 N &\cong R/\mathfrak{m}^k \otimes N = N/\mathfrak{m}^k N. \end{aligned}$$

*Hence, if  $N$  is a bounded  $\mathfrak{m}$ -torsion module then it is  $L$ -complete.*

*Proof.* See [7, proposition A.4]. □

A module  $M$  is said to be  $L$ -complete if  $\eta: M \rightarrow L_0M$  is an isomorphism. The subcategory of  $L$ -complete modules  $\widehat{\mathcal{M}} \subseteq \mathcal{M}$  is a full subcategory and the functor  $L_0: \mathcal{M} \rightarrow \widehat{\mathcal{M}}$  is left adjoint to the inclusion  $\widehat{\mathcal{M}} \rightarrow \mathcal{M}$ . The category  $\widehat{\mathcal{M}}$  has projectives, namely the *pro-free modules* which have the form

$$L_0F = F_{\mathfrak{m}}^{\widehat{}}$$

for some free  $R$ -module  $F$ . Thus  $\widehat{\mathcal{M}}$  has enough projectives and we can do homological algebra to define derived functors of right exact functors.

By [7, theorem A.6(e)], the category  $\widehat{\mathcal{M}}$  is abelian and has limits and colimits which are obtained by passing to  $\mathcal{M}$ , taking (co)limits there and applying  $L_0$ . For the latter there are non-trivial derived functors which by [6] satisfy

$$\operatorname{colim}_{\widehat{\mathcal{M}}}^s = L_s \operatorname{colim}_{\mathcal{M}},$$

so  $\operatorname{colim}^s$  is trivial for  $s > n$ . In fact, for a coproduct  $\coprod_{\alpha} M_{\alpha}$  with  $M_{\alpha} \in \widehat{\mathcal{M}}$ , we also have

$$L_n\left(\coprod_{\alpha} M_{\alpha}\right) = 0.$$

The category  $\widehat{\mathcal{M}}$  has a symmetric monoidal structure coming from the tensor product in  $\mathcal{M}$ . For  $M, N \in \mathcal{M}$ , let

$$M \widehat{\otimes} N = L_0(M \otimes N) \in \widehat{\mathcal{M}}.$$

Note that we also have

$$M \widehat{\otimes} N \cong L_0(L_0M \otimes L_0N).$$

As in [7], we find that  $(\widehat{\mathcal{M}}, \widehat{\otimes})$  is a symmetric monoidal category.

For any  $R$ -module  $M$ , there are natural homomorphisms

$$\widehat{R} \otimes L_0M \rightarrow L_0(R \otimes M) \rightarrow L_0M,$$

so we can view  $\widehat{\mathcal{M}}$  as a subcategory of  $\mathcal{M}_{\widehat{R}}$ ; since  $\widehat{R}$  is a flat  $R$ -algebra, for many purposes it is better to think of  $\widehat{\mathcal{M}}$  this way. For example, the functor  $L_0 = L_0^{\mathfrak{m}}$  on  $\mathcal{M}$  can be expressed as

$$L_0^{\mathfrak{m}}M \cong L_0^{\widehat{\mathfrak{m}}}(\widehat{R} \otimes M),$$

where  $L_0^{\widehat{\mathfrak{m}}}$  is the derived functor on the category of  $\widehat{R}$ -modules  $\mathcal{M}_{\widehat{R}}$  associated to completion with respect to the induced ideal  $\widehat{\mathfrak{m}} \triangleleft \widehat{R}$ . Finitely generated modules over  $\widehat{R}$  (which always lie in  $\widehat{\mathcal{M}}$ ) are completions of finitely generated  $R$ -modules.

There is an analogue of Nakayama's Lemma provided by [7, theorem A.6(d)].

**Proposition 1.2.** *For  $M \in \widehat{\mathcal{M}}$ ,*

$$M = \mathfrak{m}M \implies M = 0.$$

This can be used to give proofs of analogues of many standard results in the theory of finitely generated modules over commutative rings. For example,

**Corollary 1.3.** *Let  $M \in \widehat{\mathcal{M}}$  and suppose that  $N \subseteq M$  is the image of a morphism  $N' \rightarrow M$  in  $\widehat{\mathcal{M}}$ . Then*

$$M = N + \mathfrak{m}M \implies N = M.$$

*Proof.* The standard argument works here since we can form  $M/N$  in  $\widehat{\mathcal{M}}$  and as  $M/N = \mathfrak{m}M/N$ , we have  $M/N = 0$ , whence  $N = M$ .  $\square$

At this point we remind the reader that over a commutative local ring, every projective module is in fact free by a result of Kaplansky [12, theorem 2.5]. The proof of our next result is similar to that of the better known but weaker result for finitely generated projectives which is a direct consequence of Nakayama's Lemma.

**Corollary 1.4.** *Let  $M \in \widehat{\mathcal{M}}$  and suppose that  $F$  is a free module for which there is an isomorphism  $F/\mathfrak{m}F \cong M/\mathfrak{m}M$ . Then there is an epimorphism  $L_0F \rightarrow M$ .*

*Proof.* The isomorphism  $F/\mathfrak{m}F \xrightarrow{\cong} M/\mathfrak{m}M$  lifts to a map  $F \rightarrow M$  that factors through

$$L_0F \rightarrow L_0M \cong M,$$

which has image  $N \subseteq M$  say. There is a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & L_0F & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ F/\mathfrak{m}F & \xrightarrow{\cong} & L_0F/\mathfrak{m}L_0F & \xrightarrow{\cong} & M/\mathfrak{m}M \end{array}$$

which shows that  $M = N + \mathfrak{m}M$ , so  $N = M$ .  $\square$

Here is another example. Let  $S \subseteq \mathfrak{m}$  and let  $M = SM$  be the submodule of  $M$  consisting of all sums of elements of the form  $sz$  for  $s \in S$  and  $z \in M$ . We say that an  $R$ -module  $M$  is  *$S$ -divisible* if for every  $x \in M$  and  $s \in S$ , there exists  $y \in M$  such that  $x = sy$ , i.e.,  $M = SM$ . Since  $R$  is an integral domain, this is consistent with Lam's definition in chapter 1§3C of [11], see also corollary (3.17)'.

**Lemma 1.5.** *Let  $M \in \widehat{\mathcal{M}}$  and let  $S \subseteq \mathfrak{m}$  be non-empty. If  $M$  is  $S$ -divisible then it is trivial. In particular, injective objects in  $\widehat{\mathcal{M}}$  are trivial.*

*Proof.* For the first statement, if  $M = SM$  then  $M \subseteq \mathfrak{m}M$  and so  $M = \mathfrak{m}M$ , therefore  $M = 0$ .

Let  $M$  be injective in the category  $\widehat{\mathcal{M}}$ . Then for each  $x \in M$  there is a homomorphism  $R \rightarrow M$  for which  $1 \mapsto x$ . This extends to a homomorphism  $L_0R = \widehat{R} \rightarrow M$ . For  $s \in S$ , there is a homomorphism  $L_0R \rightarrow L_0R$  induced from multiplication by  $s$ . By injectivity there is an extension to a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L_0R & \xrightarrow{s} & L_0R \\ & & \downarrow & \swarrow \text{dotted} & \\ & & M & & \end{array}$$

so  $M$  is  $S$ -divisible.  $\square$

We will find it useful to know about some basic functors on  $\widehat{\mathcal{M}}$  and their derived functors.

Let  $N$  be an  $L$ -complete  $R$ -module. As the functors  $N \otimes (-): \widehat{\mathcal{M}} \rightarrow \mathcal{M}$  and  $L_0: \mathcal{M} \rightarrow \widehat{\mathcal{M}}$  are right exact, so is the endofunctor of  $\widehat{\mathcal{M}}$

$$M \mapsto N \widehat{\otimes} M = L_0(N \otimes M).$$

Therefore we can use resolutions by projective objects (i.e., pro-free  $L$ -complete modules) to form the left derived functors, which we will denote  $\widehat{\mathrm{Tor}}_s^R(N, -)$ , where

$$\widehat{\mathrm{Tor}}_0^R(N, M) = N \widehat{\otimes} M.$$

If  $P$  is pro-free then by definition,  $\widehat{\mathrm{Tor}}_s^R(N, P) = 0$  for  $s > 0$ . On the other hand,  $\widehat{\mathrm{Tor}}_s^R(P, -)$  need not be the trivial functor (see Appendix B). This shows that  $\widehat{\mathrm{Tor}}_s^R(-, -)$  is not a balanced bifunctor, *i.e.*, in general

$$\widehat{\mathrm{Tor}}_s^R(N, M) \not\cong \widehat{\mathrm{Tor}}_s^R(M, N).$$

By Proposition 1.1, for a finitely generated  $R$ -module  $N_0$ ,  $L_0N_0$  is a finitely generated  $\widehat{R}$ -module which induces the left exact functor

$$M \mapsto L_0(N_0 \otimes M) \cong N_0 \otimes M.$$

For  $M \in \widehat{\mathcal{M}}$ , we can choose consider a free resolution in  $\mathcal{M}$ ,

$$F_* \longrightarrow M \rightarrow 0.$$

Recalling that  $L_0M \cong M$  and  $L_sM = 0$  for  $s > 0$ , the homology of  $L_0F_*$  is

$$H_*(L_0F_*) = L_0M \cong M,$$

hence we have a resolution of  $M$  by pro-free modules

$$L_0F_* \longrightarrow M \rightarrow 0.$$

Then

$$\widehat{\mathrm{Tor}}_*^R(L_0N_0, M) = H_*(L_0(N_0 \otimes F_*)).$$

But now we have

$$L_0(N_0 \otimes F_*) \cong N_0 \otimes L_0F_* = N_0 \otimes (F_*)_{\mathfrak{m}}^{\widehat{}}.$$

When  $N$  is a finitely generated  $\mathfrak{m}$ -torsion module, we have  $L_0N = N$  and

$$L_0(N \otimes F_*) \cong N \otimes F_*,$$

therefore

$$(1.2) \quad \widehat{\mathrm{Tor}}_*^R(N, M) = \mathrm{Tor}_*^R(N, M).$$

Now take a free resolution

$$P_* \longrightarrow N \rightarrow 0$$

with each  $P_s$  finitely generated. Then

$$\widehat{\mathrm{Tor}}_*^R(M, N) = H_*(L_0(M \otimes P_*)) \cong H_*(M \otimes P_*) = \mathrm{Tor}_*^R(M, N),$$

hence

$$(1.3) \quad \widehat{\mathrm{Tor}}_*^R(M, N) \cong \mathrm{Tor}_*^R(M, N).$$

Combining (1.2) and (1.3), we obtain the following restricted result on  $\widehat{\mathrm{Tor}}_*^R(-, -)$  as a balanced functor. As far as we can determine, there is no general analogue of this for arbitrary  $L$ -complete modules  $N$  which are finitely generated as  $\widehat{R}$ -modules.

**Proposition 1.6.** *Let  $M, N$  be  $L$ -complete  $R$ -modules, where  $N$  is a finitely generated  $\mathfrak{m}$ -torsion module. Then*

$$\widehat{\mathrm{Tor}}_*^R(M, N) \cong \mathrm{Tor}_*^R(M, N) \cong \widehat{\mathrm{Tor}}_*^R(N, M).$$

When  $N$  is a finitely generated  $\mathfrak{m}$ -torsion module, we may also consider the composite functor  $\mathcal{M} \longrightarrow \widehat{\mathcal{M}}$  for which

$$M \mapsto L_0(N \otimes M).$$

Since

$$L_0(N \otimes M) = N \otimes L_0M = N \otimes M,$$

this functor has for its left derived functors  $\mathrm{Tor}_*^R(N, -)$  and there is an associated composite functor spectral sequence.

**Proposition 1.7.** *Let  $N$  be a finitely generated  $\mathfrak{m}$ -torsion module. Then for each  $R$ -module  $M$ , there is a natural first quadrant spectral sequence*

$$E_{s,t}^2 = \widehat{\mathrm{Tor}}_s^R(N, L_tM) = \mathrm{Tor}_s^R(N, L_tM) \implies \mathrm{Tor}_{s+t}^R(N, M).$$

*Proof.* Let  $P_* \longrightarrow N \rightarrow 0$  and  $Q_* \longrightarrow M \rightarrow 0$  be free resolutions. Since  $R$  is Noetherian, we can assume that each  $P_s$  is finitely generated, so

$$L_0(P_* \otimes Q_*) \cong P_* \otimes L_0Q_*.$$

Taking first homology, then second homology, and using (1.2) together with the fact that each  $L_0Q_t$  is projective in  $\widehat{\mathcal{M}}$ , we obtain

$$\begin{aligned} H_*^{II} H_*^I(P_* \otimes L_0Q_*) &= H_*^{II} \mathrm{Tor}_*^R(N, L_0Q_*) \\ &= H_*^{II} \widehat{\mathrm{Tor}}_*^R(N, L_0Q_*) \\ &= H_*^{II} \widehat{\mathrm{Tor}}_0^R(N, L_0Q_*) \\ &= H_*^{II}(N \otimes L_0Q_*) \\ &= H_*^{II}(N \otimes Q_*) \\ &= \mathrm{Tor}_*^R(N, M). \end{aligned}$$

The resulting spectral sequence collapses at its  $E^2$ -term. Taking second homology then first homology we obtain

$$\begin{aligned} H_*^I H_*^{II}(P_* \otimes L_0Q_*) &= H_*^I(P_* \otimes L_*M) \\ &= \mathrm{Tor}_*^R(N, L_*M). \end{aligned}$$

This is the  $E^2$ -term of a spectral sequence converging to  $\mathrm{Tor}_*^R(N, M)$  as claimed.  $\square$

**Lemma 1.8.** *Let  $M$  be a flat  $R$ -module. Then*

$$L_sM = \begin{cases} M_{\mathfrak{m}}^{\wedge} & \text{if } s = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and  $L_0M$  is pro-free.

*Proof.* For each  $s \geq 0$ , the exact sequence of (1.1) and the flatness of  $M$  yield

$$L_sM = \begin{cases} M_{\mathfrak{m}}^{\wedge} & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The spectral sequence of Proposition (1.7) with  $N = R/\mathfrak{m}$  degenerates so that for each  $s > 0$  we obtain

$$\mathrm{Tor}_s^R(R/\mathfrak{m}, L_0M) = \mathrm{Tor}_s^R(R/\mathfrak{m}, M) = 0,$$

hence  $L_0M$  is pro-free by [7, theorem A.9(3)].  $\square$

If  $M$  is a finitely generated  $R$ -module then it has bounded  $\mathfrak{m}$ -torsion, hence by [5, theorem 1.9],  $L_0M = M_{\mathfrak{m}}^{\wedge}$  and  $L_sM = 0$  for  $s > 0$ . More generally, if  $F$  is a free module, then  $F \otimes M$  has bounded  $\mathfrak{m}$ -torsion, so

$$L_s(F \otimes M) = \begin{cases} F_{\mathfrak{m}}^{\wedge} \otimes M & \text{if } s = 0, \\ 0 & \text{if } s \neq 0. \end{cases}$$

If we choose a basis for  $F$ , we can write  $F = \bigoplus_{\alpha} R$ , and the last observation amounts to the vanishing of the higher derived functors of the coproduct functor

$$\widehat{\mathcal{M}}_{\text{fg}} \longrightarrow \widehat{\mathcal{M}}; \quad M \mapsto L_0\left(\bigoplus_{\alpha} M_{\alpha}\right)$$

defined on the subcategory of completions of finitely generated modules (which is the same as the subcategory of finitely generated  $\widehat{R}$ -modules). Now for any resolution  $P_* \longrightarrow M \rightarrow 0$  of a finitely generated module  $M$  by finitely generated projectives, we have

$$L_0(F \otimes P_*) \cong L_0F \otimes P_*.$$

The left hand side has as its homology the above derived functors, so

$$H_*L_0(F \otimes P_*) = L_0(F \otimes M) = L_0F \otimes M,$$

while the right hand side has homology

$$H_*(L_0F \otimes P_*) = \text{Tor}_*^R(L_0F, M).$$

So for  $s > 0$ ,  $\text{Tor}_s^R(L_0F, M) = 0$ .

For  $P \in \widehat{\mathcal{M}}$ , the functor on  $\widehat{\mathcal{M}}$  given by  $M \mapsto P \widehat{\otimes} M$  is right exact. We say that  $P$  is *L-flat* if the functor  $P \widehat{\otimes} (-)$  is exact on  $\widehat{\mathcal{M}}$ . However, the *L-flat* modules are easily identified, at least when  $n = \dim R = 1$ , because of the following result.

**Proposition 1.9.** *Let  $P \in \widehat{\mathcal{M}}$  be L-flat. Then  $P$  is pro-free.*

*Proof.* The proof is similar to that of Corollary 1.4 and is based on a standard argument for finitely presented flat modules over a local ring. Choose a free  $R$ -module  $F$  for which  $F/\mathfrak{m}F \cong P/\mathfrak{m}P$ . If  $f: F \longrightarrow P$  is a homomorphism covering this isomorphism, there is an extension to a homomorphism  $\widehat{f}: F_{\mathfrak{m}}^{\wedge} \longrightarrow P$ . Then we have

$$\text{im } \widehat{f} + \mathfrak{m}P = P$$

and so

$$\mathfrak{m}(P/\text{im } \widehat{f}) = P/\text{im } \widehat{f},$$

hence  $\text{im } \widehat{f} = P$  by Nakayama's Lemma.

Let  $K = \ker \widehat{f}$ . Tensoring the exact sequence

$$0 \rightarrow K \longrightarrow F_{\mathfrak{m}}^{\wedge} \longrightarrow P \rightarrow 0$$

with  $R/\mathfrak{m}$ , by flatness of  $P$  we obtain the exact sequence

$$0 \rightarrow K/\mathfrak{m}K \longrightarrow F/\mathfrak{m}F \xrightarrow{\cong} P/\mathfrak{m}P \rightarrow 0$$

so  $K/\mathfrak{m}K = 0$ . Hence  $K = 0$  by Nakayama's Lemma.  $\square$

In general, tensoring with a pro-free module need not be left exact on  $\widehat{\mathcal{M}}$  as is shown by an example in Appendix B. In particular, when  $n > 1$ , infinitely generated pro-free modules may not always be flat. Instead we can restrict attention to  $L$ -flatness on restricted classes of  $L$ -complete modules. We say that  $P$  is *weakly  $L$ -flat* if the functor

$$\widehat{\mathcal{M}}_{\text{bt}} \longrightarrow \widehat{\mathcal{M}}; \quad M \mapsto P \widehat{\otimes} M$$

is exact on the subcategory  $\widehat{\mathcal{M}}_{\text{bt}}$  of bounded  $\mathfrak{m}$ -torsion modules. Then if  $Q$  is a flat module,  $L_0 Q$  is weakly  $L$ -flat since for any  $N \in \widehat{\mathcal{M}}_{\text{bt}}$ ,

$$L_0 Q \widehat{\otimes} N \cong L_0(Q \otimes N) \cong Q \widehat{\otimes} N.$$

## 2. $L$ -COMPLETE HOPF ALGEBROIDS

To ease notation, from now on we assume that  $(R, \mathfrak{m})$  is a commutative Noetherian regular local ring which is  $\mathfrak{m}$ -adically complete, *i.e.*,  $R = \widehat{R} = R_{\widehat{\mathfrak{m}}}$ . We assume that  $R$  is an algebra over some chosen local subring  $(\mathbb{k}_0, \mathfrak{m}_0)$  so that the inclusion map is local, *i.e.*,  $\mathfrak{m}_0 = \mathbb{k}_0 \cap \mathfrak{m}$ . We write  $\mathbb{k} = R/\mathfrak{m}$  for the residue field.

Let  $\Gamma \in \mathcal{M}_{\mathbb{k}_0}$ . We need to assume extra structure on  $\Gamma$  to define the notion of an  *$L$ -complete Hopf algebroid*. Unfortunately this is quite complicated to describe.

A (non-unital) *ring object*  $A \in \mathcal{M}_{\mathbb{k}_0}$  is equipped with a product morphism  $\varphi: A \otimes_{\mathbb{k}_0} A \longrightarrow A$  which is associative, *i.e.*, the following diagram commutes.

$$\begin{array}{ccc} A \otimes_{\mathbb{k}_0} A \otimes_{\mathbb{k}_0} A & \xrightarrow{\text{id} \otimes \varphi} & A \otimes_{\mathbb{k}_0} A \\ \text{id} \otimes \varphi \downarrow & & \downarrow \varphi \\ A \otimes_{\mathbb{k}_0} A & \xrightarrow{\varphi} & A \end{array}$$

It is commutative if

$$\begin{array}{ccc} A \otimes_{\mathbb{k}_0} A & \xrightarrow{\text{switch}} & A \otimes_{\mathbb{k}_0} A \\ & \searrow \varphi & \swarrow \varphi \\ & A & \end{array}$$

also commutes. An  *$R$ -unit* for  $\varphi$  is a  $\mathbb{k}_0$ -algebra homomorphism  $\eta: R \longrightarrow A$ .

**Definition 2.1.** A ring object is  *$R$ -biunital* if it has two units  $\eta_L, \eta_R: R \longrightarrow A$  which extend to give a morphism  $\eta_L \otimes \eta_R: R \otimes_{\mathbb{k}_0} R \longrightarrow A$ .

To distinguish between the two  $R$ -module or  $R$ -module structures on  $A$ , we will write  ${}_R A$  and  $A_R$ . When discussing  $A_R$  we will emphasise the use of the right module structure whenever it occurs. In particular, from now on tensor products over  $R$  are to be interpreted as bimodule tensor products  ${}_R \otimes_R$ , even though we often write  $\otimes$ .

**Definition 2.2.** An  $R$ -biunital ring object  $A$  is  *$L$ -complete* if  $A$  is  $L$ -complete as both a left and a right  $R$ -module.

**Definition 2.3.** Suppose that  $\Gamma$  is an  $L$ -complete commutative  $R$ -biunital ring object with left and right units  $\eta_L, \eta_R: R \longrightarrow \Gamma$ , and has the following additional structure:

- a *counit*: a  $\mathbb{k}_0$ -algebra homomorphism  $\varepsilon: \Gamma \longrightarrow R$ ;
- a *coproduct*: a  $\mathbb{k}_0$ -algebra homomorphism  $\psi: \Gamma \longrightarrow \Gamma \widehat{\otimes} \Gamma = \Gamma_R \widehat{\otimes}_R \Gamma$ ;
- an *antipode*: a  $\mathbb{k}_0$ -algebra homomorphism  $\chi: \Gamma \longrightarrow \Gamma$ .



Then  $\Gamma$  is an *L-complete Hopf algebroid* if

- with this structure,  $\Gamma$  becomes a cogroupoid object,
- if  $\Gamma$  is pro-free as a left (or equivalently as a right)  $R$ -module,
- the ideal  $\mathfrak{m} \triangleleft R$  is *invariant*, i.e.,  $\mathfrak{m}\Gamma = \Gamma\mathfrak{m}$ .

We often denote such a pair by  $(R, \Gamma)$  when the structure maps are clear.

The cogroupoid condition is essentially the same as that spelt out in [13, definition A1.1.1] by interpreted in the the context of  $L$ -complete bimodules. In particular we have a relationship between the two notions of  $L$ -completeness for  $\Gamma$  since the antipode  $\chi$  induces an isomorphism of  $R$ -modules  $\chi: {}_R\Gamma \cong \Gamma_R$ . The pro-freeness condition is a disguised version of flatness required to do homological algebra, and

**Definition 2.4.** Let  $(R, \Gamma)$  be an  $L$ -complete Hopf algebroid and let  $M \in \widehat{\mathcal{M}}$ . Then an  $R$ -module homomorphism  $\rho: M \rightarrow \Gamma \widehat{\otimes} M$  makes  $M$  into a *left  $(R, \Gamma)$ -comodule* or  $\Gamma$ -comodule if the following diagrams commute.

$$\begin{array}{ccc} M & \xrightarrow{\rho} & \Gamma \widehat{\otimes} M \\ \rho \downarrow & & \downarrow \psi \otimes \text{id} \\ \Gamma \widehat{\otimes} M & \xrightarrow{\rho \otimes \text{id}} & \Gamma \widehat{\otimes} \Gamma \widehat{\otimes} M \end{array} \quad \begin{array}{ccc} M & \xrightarrow{\rho} & \Gamma \widehat{\otimes} M \\ \searrow \cong & & \downarrow \varepsilon \otimes \text{id} \\ & & R \widehat{\otimes} M \end{array}$$

There is a similar definition of a *right  $\Gamma$ -comodule*.

Let  $(R, \Gamma)$  be an  $L$ -complete Hopf Algebroid. Then given a morphism of  $\Gamma$ -comodules  $\theta: M \rightarrow N$ , there is a commutative diagram of solid arrows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \theta & \longrightarrow & M & \xrightarrow{\theta} & N \\ & & \downarrow \text{dotted} & & \downarrow \psi & & \downarrow \psi \\ & & \Gamma \widehat{\otimes} \ker \theta & \longrightarrow & \Gamma \widehat{\otimes} M & \xrightarrow{\text{id} \otimes \theta} & \Gamma \widehat{\otimes} N \end{array}$$

but if  $\text{id} \otimes \theta$  is not a monomorphism then the dotted arrow may not exist or be unique. If  $\Gamma \widehat{\otimes}(-)$  always preserved exactness then this would not present a problem, but this is not so easily ensured in great generality.

If  $\Gamma$  is pro-free, then as already noted,  $\Gamma \widehat{\otimes}(-)$  is exact on the categories  $\widehat{\mathcal{M}}_{\text{bt}}$  and  $\widehat{\mathcal{M}}_{\text{fg}}$ , so in each of these contexts the above diagram always has a completion by a unique dotted arrow. Therefore the categories of  $\Gamma$ -comodules in  $\widehat{\mathcal{M}}_{\text{bt}}$  and  $\widehat{\mathcal{M}}_{\text{fg}}$  are abelian since they have kernels and all the other axioms are satisfied.

**Example 2.5.** Let  $(R, \Gamma)$  be a flat Hopf algebroid over the commutative Noetherian regular local ring  $R$ , and assume that  $\mathfrak{m}\Gamma = \Gamma\mathfrak{m}$ . By Lemma 1.8,

$$L_0({}_R\Gamma) = \Gamma_{\mathfrak{m}}^{\widehat{\phantom{x}}} = L_0(\Gamma_R),$$

where  $\Gamma_{\mathfrak{m}}^{\widehat{\phantom{x}}}$  denotes the completion with respect to the ideal  $\mathfrak{m}\Gamma$  which equals  $\Gamma\mathfrak{m}$ .

**Definition 2.6.** Let  $(R, \Gamma)$  be a Hopf algebroid over a local ring  $(R, \mathfrak{m})$  or an  $L$ -complete Hopf algebroid.

- The maximal ideal  $\mathfrak{m} \triangleleft R$  is *invariant* if  $\mathfrak{m}\Gamma = \Gamma\mathfrak{m}$ . More generally, a subideal  $I \subseteq \mathfrak{m}$  is *invariant* if  $I\Gamma = \Gamma I$ .

- An  $(R, \Gamma)$ -comodule  $M$  is *discrete* if for each element  $x \in M$ , there is a  $k \geq 1$  for which  $\mathfrak{m}^k x = \{0\}$ ; if  $M$  is also finitely generated as an  $R$ -module, then  $M$  is discrete if and only if there is a  $k_0$  such that  $\mathfrak{m}^{k_0} M = \{0\}$ .
- An  $(R, \Gamma)$ -comodule  $M$  is *finitely generated* if it is finitely generated as an  $R$ -module.

If  $M$  is a  $(R, \Gamma)$ -comodule, then for any invariant ideal  $I$ ,  $IM \subseteq M$  is a subcomodule.

If  $(R, \Gamma)$  be a (possibly  $L$ -complete) Hopf algebroid for which  $\mathfrak{m}$  is invariant, then  $(\mathbb{k}, \Gamma/\mathfrak{m}\Gamma)$  is a Hopf algebroid over the residue field  $\mathbb{k}$ . If a  $\Gamma$ -comodule is annihilated by  $\mathfrak{m}$  then it is also a  $\Gamma/\mathfrak{m}\Gamma$ -comodule.

### 3. UNIPOTENT HOPF ALGEBROIDS

We start by recalling the notion of a *unipotent* Hopf algebra  $H$  over a field  $\mathbf{k}$  which can be found in [15]. This means that every  $H$ -comodule  $V$  which is a finite dimensional  $\mathbf{k}$ -vector space has primitive elements, or equivalently (by the Jordan-Hölder theorem) it admits a composition series, *i.e.*, a finite length filtration by subcomodules

$$(3.1) \quad V = V_m \supset V_{m-1} \supset \cdots \supset V_1 \supset V_0 = \{0\}$$

with irreducible quotient comodules  $V_k/V_{k+1} \cong \mathbf{k}$ . In particular, notice that  $\mathbf{k}$  is the only finite dimensional irreducible  $H$ -comodule. Reinterpreting  $H$ -comodules as  $H^*$ -modules where  $H^*$  is the  $\mathbf{k}$ -dual of  $H$ , this implies that  $H^*$  is a local ring, *i.e.*, its augmentation ideal is its unique maximal left ideal and therefore agrees with its Jacobson radical.

Now given a Hopf algebroid  $(R, \Gamma)$  over local ring  $(R, \mathfrak{m})$  with residue field  $\mathbb{k} = R/\mathfrak{m}$  and invariant maximal ideal  $\mathfrak{m}$ , the resulting Hopf algebroid  $(\mathbb{k}, \Gamma/\mathfrak{m}\Gamma)$  need not be a Hopf algebra. However, we can still make the following definition.

**Definition 3.1.** Let  $(\mathbf{k}, \Sigma)$  be a Hopf algebroid over a field  $\mathbf{k}$ . Then  $\Sigma$  is *unipotent* if every non-trivial finite dimensional  $\Sigma$ -comodule  $V$  has non-trivial primitives. Hence  $\mathbf{k}$  is the only irreducible  $\Sigma$ -comodule and every finite dimensional comodule admits a composition series as in (3.1).

In the next result we make use of Definition 2.6.

**Theorem 3.2.** Let  $(R, \Gamma)$  be a Hopf algebroid over a Noetherian local ring  $(R, \mathfrak{m})$  for which  $\mathfrak{m}$  is invariant and suppose that  $(\mathbb{k}, \Gamma/\mathfrak{m}\Gamma)$  is a unipotent Hopf algebroid over the residue field  $\mathbb{k}$ . Let  $M$  be a non-trivial finitely generated discrete  $(R, \Gamma)$ -comodule. Then  $M$  admits a finite-length filtration by subcomodules

$$M = M_\ell \supset M_{\ell-1} \supset \cdots \supset M_1 \supset M_0 = \{0\}$$

with trivial quotient comodules  $M_k/M_{k+1} \cong \mathbb{k}$ .

See [1] for a precursor of this result. We will refer to such filtrations as *Landweber filtrations*.

*Proof.* The proof is similar to that used in [1]. The idea is to consider the descending sequence

$$M \supseteq \mathfrak{m}M \supseteq \cdots \supseteq \mathfrak{m}^k M \supseteq \cdots$$

which must eventually reach 0. So for some  $k_0$ ,  $\mathfrak{m}^{k_0} M = 0$  and  $\mathfrak{m}^{k_0-1} M \neq 0$ . The subcomodule

$$\mathfrak{m}^{k_0-1} M \cong \mathfrak{m}^{k_0-1} M / \mathfrak{m}^{k_0} M$$

becomes a comodule over  $(\mathbb{k}, \Gamma/\mathfrak{m}\Gamma)$  and so it has non-trivial primitives since  $(\mathbb{k}, \Gamma/\mathfrak{m}\Gamma)$  is unipotent, and these are also primitives with respect to  $\Gamma$ . Considering the quotient  $M/PM$ , where

$PM$  is the submodule of primitives, now we can use induction on the length of a composition series to construct the required filtration. Note that since  $R$  is local, its only irreducible module is its residue field  $\mathbb{k}$  which happens to be a comodule.  $\square$

Ravenel [13] introduced the associated Hopf algebra  $(A, \Gamma')$  to a Hopf algebroid  $(A, \Gamma)$ . When the coefficient ring  $A$  is a field, the relationship between comodules over these two Hopf algebroids turns out to be tractable as we will soon see.

Our next result provides a criterion for establishing when a Hopf algebra is unipotent. We write  $\otimes$  for  $\otimes_{\mathbf{k}}$ .

**Lemma 3.3.** *Let  $(H, \mathbf{k})$  be a Hopf algebra over a field.*

(a) *Suppose that*

$$\mathbf{k} = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n \subseteq \cdots \subseteq H$$

*is an increasing sequence of  $\mathbf{k}$ -subspaces for which  $H = \bigcup_n H_n$  and*

$$\psi H_n \subseteq H_0 \otimes H_n + H_1 \otimes H_{n-1} + \cdots + H_n \otimes H_0.$$

*Then  $H$  is unipotent. Furthermore, the  $H_n$  can be chosen to be maximal and satisfy*

$$H_r H_s \subseteq H_{r+s}$$

*for all  $r, s$ .*

(b) *Suppose that  $H$  has a filtration as in (a) and let  $W$  be a non-trivial left  $H$ -comodule which is finite dimensional over  $\mathbf{k}$  and has coaction  $\rho: W \rightarrow H \otimes W$ . Defining*

$$W_k = \rho^{-1}(H_k \otimes W) \subseteq W,$$

*we obtain an exhaustive strictly increasing filtration of  $W$  by subcomodules*

$$\{0\} = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_\ell = W.$$

*Proof.* (a) This is part of the theorem of [15, section 8.3]. The proof actually shows that the filtration by subspaces defined in (b) is strictly increasing.

(b) The fact that  $W_k$  is a subcomodule follows by comparing the two sides of the equation

$$(\text{Id} \otimes \rho)\rho(w) = \rho(\psi \otimes \text{Id})\rho(w)$$

for  $w \in W_k$ . Thus if we choose a basis  $t_1, \dots, t_d$  for  $H_k$  and write

$$\rho(w) = \sum_j t_j \otimes w_j$$

for some  $w_i \in W_k$ , then for suitable  $a_{i,r,s} \in \mathbf{k}$  we have

$$\psi(t_i) = \sum_{r,s} a_{i,r,s} t_r \otimes t_s$$

since  $\psi(t_i) \in \sum_i H_i \otimes H_{k-i} \subseteq H_k \otimes H_k$ . Therefore

$$\sum_i t_i \otimes \rho(w_i) = \sum_{j,r,s} a_{j,i,s} t_r \otimes t_s \otimes w_j,$$

and comparing the coefficients of the left hand  $t_i$ , we obtain

$$\rho(w_i) = \sum_{j,s} a_{j,i,s} t_s \otimes w_j \in H_k \otimes W.$$

This shows that each  $w_i \in W_k$ , so the coproduct restricted to  $W_k$  satisfies  $\rho W_k \subseteq H \otimes W_k$ .  $\square$

**Example 3.4.** Let  $p$  be an odd prime and let

$$\mathcal{P}_* = \mathbb{F}_p[\zeta_k : k \geq 1]$$

be the (graded) polynomial sub-Hopf algebra of the mod  $p$  dual Steenrod algebra  $\mathcal{A}_*$  with coaction

$$\psi\zeta_n = \sum_{r=0}^n \zeta_r \otimes \zeta_{n-r}^{p^r},$$

where  $\zeta_0 = 1$ . Then  $(\mathcal{P}_*, \mathbb{F}_p)$  is unipotent since the subspaces

$$\mathcal{P}(n)_* = \mathbb{F}_p[\zeta_k : 1 \leq k \leq n]$$

satisfy the conditions of Lemma 3.3. This shows that  $\mathcal{P}^*$  is a local ring.

If  $p = 2$ , this also applies to the mod 2 dual Steenrod algebra and implies that  $\mathcal{A}^*$  is a local ring.

For details on the next example, see the books by Ravenel and Wilson [13, 16]. Unfortunately the sub-Hopf algebra  $K(n)_*(E(n))$  is commonly denoted  $K(n)_*K(n)$  in the earlier literature, but at the behest of the referee we refrain from perpetuating that usage.

**Example 3.5.** Let  $p$  be an odd prime and let  $K(n)$  be the  $n$ -th  $p$ -primary Morava  $K$ -theory. Then  $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$ , with  $v_n \in K(n)_{2(p^n-1)}$ . There is a graded Hopf algebra over  $K(n)_*$ ,

$$\Gamma(n)_* = K(n)_*(E(n)) = K(n)_*[t_k : k \geq 1] / (v_n t_\ell^{p^n} - v_n^{p^\ell} t_\ell : \ell \geq 1),$$

where  $t_k \in \Gamma(n)_{2(p^k-1)}$  and  $E(n)$  is a Johnson-Wilson spectrum. In fact  $\Gamma(n)_*$  is a proper sub-Hopf algebra of  $K(n)_*(K(n))$ . Using standard formulae, it follows that the  $K(n)_*$ -subspaces

$$\Gamma(n, m)_* = K(n)_*(t_1, \dots, t_m) \subseteq \Gamma_*$$

satisfy the conditions of Lemma 3.3, therefore  $\Gamma(n, m)_*$  is unipotent. When  $p = 2$ , the Hopf algebra  $\Gamma(n)_*$  is also unipotent even though  $K(n)$  is not homotopy commutative.

Here is a major source of examples which includes the algebraic ingredients used in [1] to prove the existence of a Landweber filtration for discrete comodules over the Hopf algebroid of Lubin-Tate theory. For two topologised objects  $X$  and  $Y$  we denote the set of all continuous maps  $X \longrightarrow Y$  by  $\text{Map}^c(X, Y)$ .

**Example 3.6.** Let  $G$  be a pro- $p$ -group and suppose that  $G$  acts continuously (in the sense that the action map  $G \times R \longrightarrow R$  is continuous) by ring automorphisms on  $R$  which are continuous with respect to the  $\mathfrak{m}$ -adic topology. Then  $(R, \text{Map}^c(G, R))$  admits the structure of an  $L$ -complete Hopf algebroid, see the Appendix of [1] for details. This structure is dual to one on the twisted group algebra  $R[G]$ . Here  $\mathfrak{m}$  is invariant. If the residue field  $\mathbb{k}$  has characteristic  $p$ , then  $(\mathbb{k}, \text{Map}^c(G, R)/\mathfrak{m})$  is the continuous dual of the pro-group ring

$$\mathbb{k}[G] = \lim_{N \triangleleft G} \mathbb{k}[G/N],$$

where  $N$  ranges over the finite index normal subgroups of  $G$ . Each finite group ring  $\mathbb{k}[G/N]$  is local since its augmentation ideal is nilpotent, hence its only irreducible module is the trivial module. From this it easily follows that the dual Hopf algebra  $(\mathbb{k}, \text{Map}^c(G, R)/\mathfrak{m})$  is unipotent.

In each of the examples we are interested in, there is a filtration

$$G = G_0 \supset G_1 \supset \dots \supset G_k \supset G_{k-1} \supset \dots$$

by finite index normal subgroups  $G_k \triangleleft G$  satisfying  $\bigcap_k G_k = \{1\}$ , and the images of the natural maps

$$\text{Map}(G/G_k, R) \longrightarrow \text{Map}^c(G, R)$$

induced by the quotient maps  $G \longrightarrow G/G_k$  define a filtration with the properties listed in Lemma 3.3(a).

#### 4. UNICURSAL HOPF ALGEBROIDS

The notion of a *unicursal* Hopf algebroid  $(A, \Psi)$  appeared in [13], see definition A1.1.11. It amounts to requiring that for the subring

$$A^\Psi = A \square_\Psi A \subseteq A \otimes_A A \cong A$$

we have

$$\Psi = A \otimes_{A^\Psi} A.$$

If  $A$  is a flat  $A^\Psi$ -algebra then  $\Psi$  is a flat  $A$ -algebra. But the requirement that  $A^\Psi$  is the equalizer of the two homomorphisms  $A \longrightarrow A \otimes_{A^\Psi} A$  is implied by faithful flatness, see the second theorem of [15, section 13.1].

Unicursal Hopf algebroids were introduced by Ravenel [13]. However, his lemma A1.1.13 has a correct statement for (b), but the statement for (a) appears to be incorrect. The proofs of (a) and (b) both appear to have minor errors or gaps. In particular the flatness of  $\Psi$  as an  $A$ -module is required. Therefore we provide a slight modification of the proof given by Ravenel. Note that we work with left rather than right comodules. The formulation and proof, clarifying our earlier version, owe much to the comments of Geoffrey Powell and the referee, particularly the relationship to descent arguments based on faithful flatness.

**Lemma 4.1.** *Let  $(A, \Psi)$  be a unicursal Hopf algebroid where  $A$  is flat over  $A^\Psi$ . Let  $M$  be a left  $\Psi$ -comodule. Then there is an isomorphism of comodules*

$$M \cong A \otimes_{A^\Psi} (A \square_\Psi M),$$

where the coaction on the right hand comodule comes from the  $\Psi$ -comodule structure on  $A$ . In particular, if  $M$  is non-trivial then the primitive subcomodule  $A \square_\Psi M$  is non-trivial.

*Proof.* The coaction on  $M$  can be viewed as a map  $\rho: M \longrightarrow A \otimes_{A^\Psi} M$ . By coassociativity,

$$(\text{Id}_A \otimes \rho)\rho = (\eta_L \otimes \text{Id}_M)\rho = (\text{Id}_A \otimes 1 \otimes \text{Id}_M)\rho,$$

hence

$$\text{Id}_A \otimes \rho - \text{Id}_A \otimes 1 \otimes \text{Id}_M: \text{im } \rho \longrightarrow A \otimes_{A^\Psi} A \otimes_{A^\Psi} M$$

must be trivial. By flatness of  $A$ ,

$$0 \rightarrow A \otimes_{A^\Psi} \ker(\rho - 1 \otimes \text{Id}_M) \longrightarrow A \otimes_{A^\Psi} M \xrightarrow[\text{Id}_A \otimes 1 \otimes \text{Id}_M]{\text{Id}_A \otimes \rho} A \otimes_{A^\Psi} A \otimes_{A^\Psi} M$$

is exact, so

$$\text{im } \rho \subseteq A \otimes_{A^\Psi} \ker(\rho - 1 \otimes \text{Id}_M) = A \otimes_{A^\Psi} (A \square_\Psi M).$$

Since  $\rho: M \longrightarrow \Psi \otimes_A M$  is split by the augmentation

$$\varepsilon \otimes \text{Id}_M: \Psi \otimes_A M \longrightarrow A \otimes_A M \cong M,$$

$\rho$  is a monomorphism. For each coaction primitive  $z \in A \square_\Psi M$  and  $a \in A$ , we have

$$\rho(az) = a \otimes z \in A \otimes_{A^\Psi} (A \square_\Psi M),$$

hence  $\text{im } \rho = A \otimes_{A^\Psi} (A \square_\Psi M)$ . So we have shown that

$$M \cong A \otimes_{A^\Psi} (A \square_\Psi M). \quad \square$$

**Remark 4.2.** The above algebra can be interpreted scheme-theoretically as follows. Given a flat morphism of affine schemes  $f: X \rightarrow Y$ ,  $X \times_Y X$  becomes a groupoid scheme with a unique morphism  $u \rightarrow v$  whenever  $f(u) = f(v)$ . Comodules for the representing Hopf algebroid are equivalent to  $\mathcal{O}_X$ -modules with descent data, and the category of such comodules is equivalent to that of  $\mathcal{O}_Y$ -modules. See [15, section 17.2] for an algebraic version of this when  $f$  is faithfully flat.

**Example 4.3.** Let  $R$  be a commutative ring and let  $G$  be a finite group which acts faithfully on  $R$  by ring automorphisms so that  $R^G \rightarrow R$  is a  $G$ -Galois extension in the sense of [3]. Thus there is an isomorphism of rings

$$(4.1) \quad R \otimes_{R^G} R \cong R \otimes_{R^G} R^G G^*,$$

where the dual group ring is  $R^G G^* = \text{Map}(G, R^G)$ . The left hand side is visibly a Hopf algebroid and as an  $R^G$ -module,  $R$  is finitely generated projective, so Lemma 4.1 applies.

Following the outline in [1], we can identify

$$R \otimes_{R^G} R^G G^* \cong R G^*$$

with the dual of the twisted group ring  $R \sharp G$  and thus it also carries a natural Hopf algebroid structure. It is easy to verify that this structure agrees with that on  $R \otimes_{R^G} R$  under (4.1).

Interpreting a  $R \otimes_{R^G} R$ -comodule  $M$  as equivalent to a  $R \sharp G$ -module, we can use the Galois theoretic isomorphism  $R \sharp G \cong \text{End}_{R^G} R$  to show that there is an isomorphism of  $R \sharp G$ -modules

$$M \cong R \otimes_{R^G} M^G,$$

and since

$$M^G \cong R \square_{R \otimes_{R^G} R} M,$$

this is a module theoretic interpretation of the comodule result of Lemma 4.1.

Now we recall some facts from [13, lemma A1.1.13] about the extension of Hopf algebroids

$$(D, \Phi) \rightarrow (A, \Gamma) \rightarrow (A, \Gamma'),$$

where  $\Gamma'$  is the Hopf algebra associated to  $\Gamma$  and  $\Phi$  is unicursal. We have the following identifications:

$$\Gamma' = A \otimes_\Phi \Gamma, \quad D = A \square_\Gamma A, \quad \Phi = A \otimes_D A.$$

The map of Hopf algebroids  $\Gamma \rightarrow \Gamma'$  is normal and

$$\Phi = \Gamma \square_{\Gamma'} A = A \square_{\Gamma'} \Gamma \subseteq \Gamma.$$

Furthermore, for any left  $\Gamma$ -comodule  $M$ ,  $A \square_{\Gamma'} M$  is naturally a left  $\Phi$ -comodule and there is an isomorphism of  $A$ -modules

$$(4.2) \quad A \square_\Gamma M \cong A \square_\Phi (A \square_{\Gamma'} M).$$

**Proposition 4.4.** *Let  $M$  be a  $\Gamma$ -comodule. If when viewed as a  $\Gamma'$ -comodule,  $M$  has non-trivial primitive  $\Gamma'$ -subcomodule  $A \square_{\Gamma'} M$ , then the primitive  $\Gamma$ -subcomodule  $A \square_\Gamma M$  is non-trivial.*

*Proof.* Combine Lemma 4.1 and (4.2).  $\square$

Our next result is immediate.

**Theorem 4.5.** *Let  $(\mathbf{k}, \Gamma)$  is a Hopf algebroid over a field. If the associated Hopf algebra  $(\mathbf{k}, \Gamma')$  is unipotent, then  $(\mathbf{k}, \Gamma)$  is unipotent.*

## 5. LUBIN-TATE SPECTRA AND THEIR HOPF ALGEBROIDS

In this section we will discuss the case of a Lubin-Tate spectrum  $E$  and its associated Hopf algebroid  $(E_*, E_*^\vee E)$ , where  $E$  denotes any of the 2-periodic spectra lying between  $\widehat{\mathcal{E}(n)}$  (by which we mean the 2-periodic version of the completed  $2(p^n - 1)$ -periodic Johnson-Wilson spectrum  $\widehat{E(n)}$ ) and  $E_n^{\text{nr}}$  discussed in [2], see especially section 7. The most important case is the ‘usual’ Lubin-Tate spectrum  $E_n$  for which

$$\pi_*(E_n) = W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]] [u^{\pm 1}],$$

but other examples are provided by the  $K(n)$ -local Galois subextension of  $E_n^{\text{nr}}$  over  $\widehat{\mathcal{E}(n)}$  in the sense of Rognes [14]. In all cases,  $E_* = \pi_*(E)$  is a local ring with maximal ideal induced from that of  $\widehat{E(n)}_*$ , and we will write  $\mathfrak{m}$  for this. The residue field  $E_*/\mathfrak{m}$  is always a graded subfield of the algebraic closure  $\bar{\mathbb{F}}_p[u, u^{-1}]$  of  $\mathbb{F}_p[u, u^{-1}]$ .

**Remark 5.1.** Since all of the spectra considered here are 2-periodic we will sometimes treat their homotopy as  $\mathbb{Z}/2$ -graded and as it is usually trivial in odd degrees, we will often focus on even degree terms. However, when discussing reductions modulo a maximal ideal, it is sometimes more useful to regard the natural periodicity as having degree  $2(p^n - 1)$  with associated  $\mathbb{Z}/2(p^n - 1)$ -grading; more precisely, we will follow the ideas of [4] and take consider gradings on  $\mathbb{Z}/2(p^n - 1)$  together with the non-trivial bilinear pairing

$$\nu: \mathbb{Z}/2(p^n - 1) \times \mathbb{Z}/2(p^n - 1) \longrightarrow \{1, -1\}; \quad \nu(\bar{i}, \bar{j}) = (-1)^{ij},$$

where  $\bar{i}$  denotes the residue class  $i \pmod{2(p^n - 1)}$ .

We will denote by  $K = E \wedge_{\widehat{E(n)}} K(n)$  the version of Morava  $K$ -theory associated to  $E$ , it is known that  $E$  is  $K$ -local in the category of  $E$ -modules and we can consider the localisation  $L_K(E \wedge E)$  for which

$$E_*^\vee E = \pi_*(L_K(E \wedge E)).$$

By [6, proposition 2.2], this localisation can be taken either with respect to  $K$  in the category of  $S$ -modules, or with respect to  $E \wedge K$  in the category of  $E$ -modules. By [2, lemma 7.6], the homotopy  $\pi_*(L_K M)$  viewed as a module over the local ring  $(E_*, \mathfrak{m})$  is  $L$ -complete.

We will write  $\text{Map}(X, Y)$  for the set of all functions  $X \longrightarrow Y$  and  $\text{Map}^c(X, Y)$  for the set of all continuous functions if  $X, Y$  are topologised.

A detailed discussion of the relevant  $K(n)$ -local Galois theory of Lubin-Tate spectra can be found in section 5.4 and chapter 8 of [14], and we adopt its viewpoint and notation. In particular,  $E_n^{\text{nr}}$  is a  $K(n)$ -local Galois extension of  $L_{K(n)}$  with profinite Galois group

$$\mathbb{G}_n^{\text{nr}} = \widehat{\mathbb{Z}} \rtimes \mathbb{S}_n,$$

where  $\mathbb{S}_n$  is the usual Morava stabiliser group which can be viewed as the full automorphism group of a height  $n$  Lubin-Tate formal group law  $F_n$  defined over  $\mathbb{F}_{p^n} \subseteq \bar{\mathbb{F}}_p$ , and also as the group of units in the maximal order of a central division algebra over  $\mathbb{Q}_p$  of Hasse invariant  $1/n$ . The  $p$ -Sylow subgroup  $\mathbb{S}_n^0 \triangleleft \mathbb{S}_n$  has index  $(p^n - 1)$  and  $\mathbb{S}_n$  is the semi-direct product

$$\mathbb{S}_n = \mathbb{F}_{p^n}^\times \rtimes \mathbb{S}_n^0.$$

The profinite group  $\widehat{\mathbb{Z}}$  acts as the Galois group

$$\mathrm{Gal}(W\bar{\mathbb{F}}_p/W\mathbb{F}_p) \cong \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}.$$

In particular, the closed subgroup  $n\widehat{\mathbb{Z}} \triangleleft \widehat{\mathbb{Z}}$  is the stabiliser of  $\mathbb{F}_{p^n}$  and  $E_n \simeq (E_n^{\mathrm{nr}})^{h(n\widehat{\mathbb{Z}})}$ ; similarly,  $\widehat{\mathcal{E}(n)} \simeq (E_n^{\mathrm{nr}})^{h\widehat{\mathbb{Z}}}$ .

Our first result is a generalisation of a well known result, see [1] for example.

**Theorem 5.2.** *For  $E$  as above, there are natural isomorphisms of  $E_0$ -algebras*

$$E_*^\vee \widehat{\mathcal{E}(n)} \cong \mathrm{Map}^c(\mathbb{S}_n, E_*).$$

Furthermore,  $\mathrm{Map}^c(\mathbb{S}_n, E_*)$  is a pro-free  $L$ -complete  $E_*$ -module.

**Theorem 5.3.** *Let  $E$  be a Lubin-Tate spectrum as above.*

- (a)  $(E_*, E_*^\vee E)$  is an  $L$ -complete Hopf algebroid.
- (b) the maximal ideal  $\mathfrak{m} \triangleleft E_*$  is invariant.
- (c)  $E_*^\vee E$  is a pro-free  $E_*$ -module.
- (d) There are isomorphisms of  $K_* = E_*/\mathfrak{m}$ -algebras

$$K_* E \cong E_*^\vee E / E_*^\vee E \mathfrak{m} \cong E_*/\mathfrak{m}[\theta_k : k \geq 1] / (\theta_\ell^{p^n} - u^{p^\ell - 1} \theta_\ell : \ell \geq 1) \otimes_{\mathbb{F}_p[u, u^{-1}]} E_*/\mathfrak{m}.$$

Now let us consider the reduction  $K_* E$  in greater detail. First note that the pair  $(K_*, K_* E)$  is a  $\mathbb{Z}$ -graded Hopf algebroid. Now

$$K_* = \mathbb{F}[u, u^{-1}],$$

where  $\mathbb{F} \subseteq \bar{\mathbb{F}}_p$  and  $|u| = 2$ . Since  $u^{p^n - 1} = v_n$  under the map  $BP \rightarrow K$  classifying a complex orientation,  $u^{p^n - 1}$  is invariant. This suggests that we might usefully change to a  $\mathbb{Z}/2(p^n - 1)$ -grading on  $K_*$ -modules by setting  $u^{p^n - 1} = 1$ . To emphasise this regrading we write  $(-)_\bullet$  rather than  $(-)_*$ . In particular,  $K_\bullet = \mathbb{F}(u)$ .

The right unit generates a second copy of  $K_\bullet$  in  $K_\bullet E$  and there is an element

$$\theta_0 = \eta_L(u)^{-1} \eta_R(u)$$

which satisfies the relation

$$\theta_0^{p^n - 1} = 1.$$

The coproduct makes  $\theta_0$  group-like,

$$\psi(\theta_0) = \theta_0 \otimes \theta_0.$$

Now it is easy to see that  $K_\bullet E$  contains the uncursal Hopf algebroid

$$(5.1) \quad K_\bullet \otimes_{\mathbb{F}_p} K_\bullet = \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}(u, \theta_0) = \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}[u, \theta_0] / (u^{p^n - 1} - 1, \theta_0^{p^n - 1} - 1),$$

where  $\theta_0, u$  have degrees  $\bar{0}, \bar{2} \in \mathbb{Z}/2(p^n - 1)$  respectively.

Ignoring the generator  $u$  and the grading, we also have the ungraded Hopf algebroid

$$(\mathbb{F}, \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}(\theta_0)) = (\mathbb{F}, \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}[\theta_0] / (\theta_0^{p^n - 1} - 1))$$

which is a subHopf algebroid of  $(\mathbb{F}, K_0 E)$ .

Since  $\mathbb{F}$  is a Galois extension of  $\mathbb{F}_p$  with Galois group a quotient of  $\widehat{\mathbb{Z}}$ , we obtain a ring isomorphism

$$\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F} \cong \mathbb{F} \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)^*.$$

If  $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$  is finite this has its usual meaning, while if  $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$  is infinite we have

$$\mathbb{F} \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)^* = \mathrm{Map}^c(\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p), \mathbb{F}).$$



Of course, if  $\text{Gal}(\mathbb{F}/\mathbb{F}_p)$  is finite this interpretation is still valid but then all maps  $\text{Gal}(\mathbb{F}/\mathbb{F}_p) \longrightarrow \mathbb{F}$  are continuous. In each case, we obtain an isomorphism of Hopf algebroids

$$(5.2) \quad \mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}(\theta_0) \cong \text{Map}(\text{Gal}(\mathbb{F}/\mathbb{F}_p) \ltimes \mathbb{F}_{p^n}^\times, \mathbb{F}).$$

Now we consider the associated Hopf algebra over the graded field  $K_\bullet$ ,

$$K_\bullet \otimes_{\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}(u, \theta_0)} K_\bullet E = K_\bullet[\theta_k : k \geq 1] / (\theta_k^{p^n} - \theta_k : k \geq 1)$$

whose zero degree part is

$$(5.3) \quad \mathbb{F}[\theta_k : k \geq 1] / (\theta_k^{p^n} - \theta_k : k \geq 1) \cong \text{Map}^c(\mathbb{S}_n^0, \mathbb{F}).$$

The right hand side fits into the framework of Example 3.6, so this Hopf algebra over  $\mathbb{F}$  is unipotent. Tensoring up with  $K_\bullet$  we have the following graded version.

**Theorem 5.4.** *The Hopf algebra  $(K_\bullet, K_\bullet \otimes_{\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}(u, \theta_0)} K_\bullet E)$  is unipotent.*

**Remark 5.5.** The identification of (5.3) can be extended to all degrees of  $K_\bullet \otimes_{\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}(u, \theta_0)} K_\bullet E$ . To make this explicit, we consider  $\text{Map}^c(\mathbb{S}_n, \mathbb{F}u^r)$  with the action of  $\text{Gal}(\mathbb{F}/\mathbb{F}_p) \ltimes \mathbb{F}_{p^n}^\times$  induced from the action on  $\mathbb{S}_n$  used in defining  $\mathbb{G}_n^{\text{nr}}$  and the  $\mathbb{F}$ -semilinear action of  $\text{Gal}(\mathbb{F}/\mathbb{F}_p) \ltimes \mathbb{F}_{p^n}^\times$  on  $\mathbb{F}u^r$  obtained by inducing up the  $r$ -th power of the natural 1-dimensional representation of  $\mathbb{F}_{p^n}^\times$ . Then

$$[K_\bullet \otimes_{\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}(u, \theta_0)} K_\bullet E]_{2r} \cong \text{Map}^c(\mathbb{S}_n, \mathbb{F}u^r)^{\text{Gal}(\mathbb{F}/\mathbb{F}_p) \ltimes \mathbb{F}_{p^n}^\times},$$

where the right hand side is the set of continuous  $\text{Gal}(\mathbb{F}/\mathbb{F}_p) \ltimes \mathbb{F}_{p^n}^\times$ -equivariant maps. This is essentially a standard identification appearing in work of Morava and others in the 1970's.

Combining Theorems 5.4 and 3.2 we obtain our final result in which revert to  $\mathbb{Z}$ -gradings.

**Theorem 5.6.** *The Hopf algebroid  $(K_*, K_* E)$  is unipotent, hence every finitely generated comodule for the  $L$ -complete Hopf algebroid  $(E_*, E_*^\vee E)$  has a Landweber filtration.*

Here it is crucial that we take proper account of the grading since the ungraded Hopf algebra  $(K_0, K_0 E)$  is not unipotent: this can be seen by considering the comodule  $K_0 S^2$  which is not isomorphic to  $K_0 S^0$ .

## APPENDIX A. REPRESENTATIONS OF GALOIS HOPF ALGEBROIDS

Twisted (or skew) group rings are standard algebraic objects. They were discussed for an audience of topologists in [1], and their duals as Hopf algebroids were discussed. For a recent reference on their modules see [10]. Here we focus on the special case of Galois extensions of fields, which is closely related to the uncursal Hopf algebroids. In particular, the uncursal Hopf algebroids associated with  $K_\bullet$  in Section 5 contain degree zero parts of this form.

Let  $\mathbf{k}$  be a field of positive characteristic  $\text{char } \mathbf{k} = p$  and let  $A$  be a (finite dimensional) commutative  $\mathbf{k}$ -algebra which is a  $G$ -Galois extension of  $\mathbf{k}$  for some finite group  $G$ , where the action of  $\gamma \in G$  on  $x \in A$  is indicated by writing  ${}^\gamma x$ . This means that

- $A^G = \mathbf{k}$ ,
- the  $A$ -algebra homomorphism

$$A \otimes_{\mathbf{k}} A \longrightarrow \prod_{\gamma \in G} A; \quad x \otimes y \mapsto (x {}^\gamma y)_{\gamma \in G}$$

is an isomorphism, where the  $A$ -algebra comes from the left hand factor of  $A$ .

The second condition is equivalent to the assertion that there is an isomorphism of  $\mathbf{k}$ -algebras

$$(A.1) \quad A \otimes_{\mathbf{k}} A \cong A \otimes_{\mathbf{k}} \mathbf{k}G^*,$$

where

$$\mathbf{k}G^* = \text{Hom}_{\mathbf{k}}(\mathbf{k}G, \mathbf{k})$$

is the dual group algebra.

The *twisted group ring*  $A\sharp G$  is the usual group ring  $AG$  as a left  $A$ -module, but with multiplication defined by

$$(a_1\gamma_1)(a_2\gamma_2) = a_1^{\gamma_1}a_2\gamma_1\gamma_2.$$

There is a natural  $\mathbf{k}$ -linear map

$$A\sharp G \longrightarrow \text{End}_{\mathbf{k}} A$$

under which  $a\gamma \in A\sharp G$  is sent to the  $\mathbf{k}$ -linear endomorphism  $x \mapsto a^{\gamma}x$ . Another consequence of the above assumptions is that this is an  $\mathbf{k}$ -algebra isomorphism, see [3].

If  $A = \ell$  is a field, then using the isomorphism of (A.1) we see that  $\ell \otimes_{\mathbf{k}} \ell$  is isomorphic to  $\ell G^*$  as an  $\ell$ -algebra. There is an associated ‘right’ action of  $\ell$  on  $\ell G^*$  given by

$$(f \cdot x)(\gamma) = {}^{\gamma}xf(\gamma)$$

for  $f \in \ell G^*$ ,  $x \in \ell$  and  $\gamma \in G$ .

A proof of the next result is sketched in [1].

**Proposition A.1.** *The pair  $(\ell, \ell G^*)$  is a Hopf algebroid.*

**Lemma A.2.** *The twisted group ring  $\ell\sharp G$  is a simple  $\mathbf{k}$ -algebra and every finite dimensional  $\ell\sharp G$ -module  $V$  is completely reducible. In particular, if  $V \neq 0$  then  $V^G \neq 0$  and there is an  $\ell$  linear isomorphism*

$$\ell \otimes_{\mathbf{k}} V^G \longrightarrow V; \quad x \otimes v \mapsto xv.$$

*Proof.* Since  $\text{End}_{\mathbf{k}} A$  is an irreducible  $\mathbf{k}$ -algebra, it has a unique simple module which agrees with  $A$  as a  $\mathbf{k}$ -module. Hence every finite dimensional module is isomorphic to a direct sum of copies of  $A$ . Since  $A^G = \mathbf{k}$ , we see that  $V^G \neq 0$ . Verifying the bijectivity of the linear map is straightforward.  $\square$

We can generalise this situation and still get similar results. For example, if  $\tilde{G}$  is a finite group with a given epimorphism  $\pi: \tilde{G} \longrightarrow G$ , then  $\ell\sharp \tilde{G}$  is semi-simple provided that  $p \nmid |\ker \pi|$ , see [10]. In fact the uncursal Hopf algebroid  $\mathbb{F} \otimes_{\mathbb{F}_p} \mathbb{F}(\theta_0)$  of (5.2) is dual to

$$\mathbb{F}\sharp(\text{Gal}(\mathbb{F}/\mathbb{F}_p) \ltimes \mathbb{F}_{p^n}^{\times}),$$

where the action of the group on  $\mathbb{F}$  is through the projection onto  $\text{Gal}(\mathbb{F}/\mathbb{F}_p)$ .

## APPENDIX B. NON-EXACTNESS OF TENSORING WITH A PRO-FREE MODULE

In [6, section 1], it was shown that in  $\widehat{\mathcal{M}}$ , coproducts need not preserve left exactness. At the suggestion of the referee, we include a more precise example showing that tensoring with a pro-free module in  $\widehat{\mathcal{M}}$  need not be a left exact functor. This material is due to the referee to whom we are grateful for the opportunity to include it. The main result is Theorem B.5, but we need several preparatory technical results.

**Lemma B.1.** *Let*

$$M_0 \xleftarrow{f} M_1 \xleftarrow{f} M_2 \xleftarrow{f} \dots$$

*be a inverse system of abelian groups for which  $\lim_n M_n = 0 = \lim_n^1 M_n$  and where each homomorphism  $M_n \rightarrow M_0$  is non-zero. Then the induced inverse system*

$$\bigoplus_{k \in \mathbb{N}} M_0 \xleftarrow{\tilde{f}} \bigoplus_{k \in \mathbb{N}} M_1 \xleftarrow{\tilde{f}} \bigoplus_{k \in \mathbb{N}} M_2 \xleftarrow{\tilde{f}} \dots$$

*satisfies  $\lim_n^1 \bigoplus_{k \in \mathbb{N}} M_n \neq 0$ .*

*Proof.* For ease of notation we write  $\bigoplus_k$  for  $\bigoplus_{k \in \mathbb{N}}$  and  $\prod_n$  for  $\prod_{n \in \mathbb{N}_0}$ , where  $\mathbb{N} = \{0\} \cup \mathbb{N}$ .

Consider the commutative square

$$\begin{array}{ccc} \prod_n \bigoplus_k M_n & \xrightarrow{d} & \prod_n \bigoplus_k M_n \\ \downarrow & & \downarrow \\ \prod_n \prod_k M_n & \xrightarrow[\cong]{d'} & \prod_n \prod_k M_n \end{array}$$

in which  $d$  is the shift map with

$$\ker d = \lim_n^1 \left( \bigoplus_k M_n \right), \quad \text{coker } d = \lim_n \left( \bigoplus_k M_n \right),$$

and similarly for  $d'$ . The vanishing of  $\lim_n^s M_n$  for  $s = 0, 1$  implies that  $d'$  is an isomorphism.

Now choose a sequence of elements  $a_n \in M_n$  with non-zero images in  $M_0$ . Define

$$b_{nk} = \begin{cases} a_n & \text{if } k = n, \\ 0 & \text{if } k \neq n, \end{cases}$$

and let  $b = (b_{n,k})$  be the resulting element of  $\prod_n \prod_k M_n$ . Defining  $c = (d')^{-1}(b)$ , we see that

$$(B.1) \quad b_{nk} = c_{nk} - f(c_{n+1,k})$$

for all  $n, k$ . Now fix  $k$  and consider the  $c_{nk}$  for  $n > k$ ; these satisfy

$$c_{nk} = f(c_{n+1,k}),$$

hence they yield an element of inverse limit of the inverse system

$$M_k \xleftarrow{f} M_{k+1} \xleftarrow{f} M_{k+2} \xleftarrow{f} \dots,$$

but

$$\lim_{n \geq k} M_n = \lim_n M_n = 0.$$

Therefore  $c_{nk} = 0$  for  $n > k$ . Using (B.1), we see that for all  $k \geq n$ ,

$$c_{nk} = f^{k-n}(a_k)$$

and in particular, in  $M_0$ ,

$$c_{0k} = f^k(a_k) \neq 0.$$

This shows that  $c \notin \prod_n \bigoplus_k M_n$  even though  $d'(c) = b \in \prod_n \bigoplus_k M_n$ . The result follows by inspection of the diagram.  $\square$

Now let  $R = \mathbb{Z}_p[[u]]$  with  $\mathfrak{m} = (p, u)$  its maximal ideal containing  $p$  and  $u$ . Let  $M$  be the  $\mathfrak{m}$ -adic completion of  $\bigoplus_n R$  which can be identified with the set of sequences  $x = (x_n) \in \prod_n R$  for which  $x_n \rightarrow 0$  in the  $\mathfrak{m}$ -adic topology. The group  $N = \prod_n (p^n, u^n)$  is a subgroup of  $M$ .

The category of  $L$ -complete modules is closed under products and contains all finitely generated  $R$ -modules, therefore  $N$  and  $M/N$  are  $L$ -complete.

**Lemma B.2.** *For each  $n \geq 1$ , the natural map  $\mathrm{Tor}_2^R(R/\mathfrak{m}^n, M/N) \rightarrow \mathrm{Tor}_2^R(R/\mathfrak{m}, M/N)$  is non-zero.*

*Proof.* Since  $R/(p^n, u^n)$  is a retract of  $N$ , it suffices to prove the result for  $R/(p^n, u^n)$  in place of  $N$ . The sequence  $p^n, u^n$  is regular in  $R$ , so for any  $R$ -module  $K$  we can compute  $\mathrm{Tor}_*^R(R/(p^n, u^n), K)$  using a Koszul resolution. In particular, if  $p^n K = 0 = u^n K$  then we have

$$\mathrm{Tor}_*^R(R/(p^n, u^n), K) = K$$

and the reduction map is the obvious epimorphism  $R/(p^n, u^n) \rightarrow R/(p, u)$ . the result follows easily from this.  $\square$

**Corollary B.3.** *The module  $L_1(\bigoplus_k M/N)$  is non-zero.*

*Proof.* For each  $s \geq 1$ , the natural short exact sequence of (1.1) and the fact that  $M/N$  is  $L$ -complete and so  $L_s M/N = 0$ , together yield

$$\lim_n \mathrm{Tor}_2^R(R/\mathfrak{m}^n, K) = 0 = \lim_n^1 \mathrm{Tor}_2^R(R/\mathfrak{m}^n, K).$$

This gives one of the hypotheses of Lemma B.1, and Lemma B.2 gives the other. Therefore

$$\lim_n^1 \mathrm{Tor}_2^R(R/\mathfrak{m}^n, M/N) \neq 0.$$

Now applying (1.1) to  $M/N$  gives  $L_1(\bigoplus_k M/N) \neq 0$ .  $\square$

**Lemma B.4.** *If the sequence  $p, u$  acts regularly on the  $R$ -module  $K$ , then  $L_s K = 0$  for  $s > 0$ .*

*Proof.* Using the exact sequence (1.1), it suffices to show that  $\mathrm{Tor}_s^R(R/\mathfrak{m}^n, K) = 0$  for all  $s > 0$ . This can be deduced from the case  $n = 1$  since  $R/\mathfrak{m}^n$  has a composition series with simple quotient terms isomorphic  $R/\mathfrak{m}$ . This case  $n = 1$  can be directly verified using the Koszul resolution.  $\square$

Here is the main result of this Appendix which complements an example of [6].

**Theorem B.5.** *The natural map  $L_0(\bigoplus_k N) \rightarrow L_0(\bigoplus_k M)$  is not injective.*

*Proof.* The short exact sequence

$$0 \rightarrow \bigoplus_k N \rightarrow \bigoplus_k M \rightarrow \bigoplus_k M/N \rightarrow 0$$

induces an exact sequence

$$L_1(\bigoplus_k M) \rightarrow L_1(\bigoplus_k M/N) \rightarrow L_0(\bigoplus_k N) \rightarrow L_0(\bigoplus_k M) \rightarrow 0.$$

The sequence  $p, u$  acts regularly on  $\bigoplus_k M$ , so Lemma B.4 shows that  $L_1(\bigoplus_k M) = 0$ , while Corollary B.3 shows that  $L_1(\bigoplus_k M/N) \neq 0$ .  $\square$

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